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LETTER TO THE EDITOR

Ultradiffusion on the fractal branching Koch curves

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Abstract. Ultradiffusion is defined on a family of fractal branching Koch curves. By an exact renormalization decimation transformation the anomalous long-time behaviour of the autocorrelation function is obtained. For the particular hierarchy we find a non-universal crossover $C_1=(m-1)/2M$ and a universal one $C_2=1/2$ as the effective temperature C is increased, where M is the maximum ramification number of the fractal. We also find that for large M, the dynamic behaviour of ultradiffusion on the fractal is independent of the effective temperature C.

It is known that one of the parameters used to characterize the topological properties of the fractal is R, the order of ramification. At a point P, R measures the smallest number of significant interactions which one must cut in order to isolate an arbitrary bounded set of points connected to P (Mandelbrot 1977, 1982). The two extreme values of R obey the inequality, $R_{max} \geqslant 2R_{min}-2$. The fractal Koch curves have finite R. When $R_{max}=R_{min}=2$, the curve is homogeneous and it is called non-branching. When $R_{max} \neq R_{min}$, the curve is inhomogeneous and it is called branching (Gefen *et al* 1983).

Since Huberman and Kerszberg (1985) presented the typically simplest one-dimensional ultradiffusion model to describe the anomalous relaxation in very different physical systems ranging from molecular diffusion (Austin *et al* 1975) to spin glasses (Sompolinsky 1981), various aspects of hierarchical structures have been investigated, consisting of electronic properties (Ceccatto *et al* 1987), vibrational spectrum (Keirstead *et al* 1988), multifractal nature (Havlin and Matan 1988, Kahng and Redner 1989, Lin and Tao 1990), etc, and several kinds of generalized models have also emerged (Maritan and Stella 1986a, Ceccatto and Riera 1986, Ceccatto and Huberman 1988, Giacometti *et al* 1988, Zheng *et al* 1989a, 1991).

It is worth mentioning that Maritan and Stella (1986a) studied the spectral properties on a fractal non-branching Koch curve with long-range interactions and found a transition from non-universal to universal anomalous spectral behaviour as the range of the forces is increased beyond a certain threshold. They also pointed out that the problems of ultradiffusion can be solved within similar mathematical frameworks and similar conclusions can be obtained.

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It has been shown that different maximum ramification number R_{max} of the fractal will result in some different properties, such as the trail problem (Zheng *et al* 1989b) on the fractal Koch curves. In this letter, ultradiffusion on a family of fractal branching Koch curves is considered and an interesting dependence of the ultradiffusion behaviour on the maximum ramification number of the fractal branching Koch curves is obtained. First of all, we show that the anomalous relaxation exponent of the autocorrelation function depends on the maximum ramification number of the fractal. Then, we find two crossovers as the effective temperature is increased in the case of the particular hierarchy. Furthermore, it is indicated that the anomalous relaxation behaviour of the autocorrelation function is independent of the effective temperature for the fractal branching Koch curves with large ramification number.

For the sake of convenience and simplicity, in this letter we first introduce our ultradiffusion model on a fractal branching Koch curve with the maximum ramification number $R_{max}=3$, and the fractal dimension $d_f=\ln 6/\ln 3$ as pictured in figure 1.

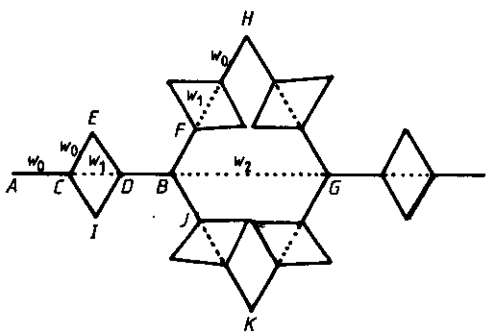

Figure 1. An ultradiffusion model defined on a branching Koch curve with maximum ramification number $R_{max}=3$ and fractal dimension $d_f=\ln 6/\ln 3$. The continuous curve represents the backbone of the fractal.

Consider a particle hopping from point to point on the curve with energy barriers distributed in a hierarchical way as indicated in the figure. The barriers are labelled by w_i ($i=0, 1, 2, 3, \ldots$), the probability that they will be crossed in unit time. Let $P_A(t)$, $P_B(t), \ldots$, be the probability of finding the particle at point A, $B, \ldots$, at time t and $\tilde{P}_A(\lambda)$, $\tilde{P}_B(\lambda), \ldots$, their corresponding Laplace transforms. Then the diffusion is described by an infinite set of master equations of the following type

$$\lambda \tilde{P}_B = w_0(\tilde{P}_D - \tilde{P}_B) + w_0(\tilde{P}_F - \tilde{P}_B) + w_0(\tilde{P}_J - \tilde{P}_B) + w_2(\tilde{P}_G - \tilde{P}_B). \tag{1}$$

Performing an exact renormalization group decimation procedure which describes the points C, D, E, I, F, $J, \ldots$, in terms of A, B, G, H and K, we obtain a new system of the same form as (1) with the recursion relations

$$\tilde{P}' = \alpha \tilde{P} \qquad w_j' = w_{j+1}/\alpha \qquad \lambda' = \beta\lambda \tag{2}$$

where

$$\alpha = (w_0 + w_1)/(3w_0 + 2w_1) \tag{3}$$

and

$$\beta = (21 w_0 + 14 w_1)/(w_0 + w_1). \tag{4}$$

In deriving the above relations we have used the condition $w_0' = w_0$ to fix the timescale and considered only the $\lambda \to 0$ limit to obtain the leading dynamical scaling behaviour of the autocorrelation function $P_0(t)$ (Maritan and Stella 1986b).

The above model and method are easily generalized to the case of the Koch curve in the same family with the arbitrary maximum ramification number $R_{\max} = M$ $(M \geqslant 3)$ and the fractal dimension $d_f = \ln 2M/\ln 3$. The corresponidng recursion relations are found

$$\tilde{P}' = \alpha_M \hat{P} \qquad w_j' = w_{j+1}/\alpha_M \qquad \lambda' = \beta_M \lambda \tag{5}$$

where

$$\alpha_M = [(M-1)w_0 + 2w_1]/(2Mw_0 + 4w_1) \tag{6}$$

and

$$\beta_M = [(M^2+M+2)(Mw_0+2w_1)]/[(M-1)w_0 + 2w_1]. \tag{7}$$

Taking into account (5)-(7) and paying attention to the inverse Laplace transform of $\tilde{P}_0(\lambda)$, the non-universal time scaling exponent x (Maritan and Stella 1986b) of the autocorrelation function is found to be

$$x = \frac{2\ln[(M^2+M+2)(Mw_0+2w_1^*)/(2Mw_0+4w_1^*)]}{\ln\{(M^2+M+2)(Mw_0+2w_1^*)/[(M-1)w_0+2w_1^*]\}} \tag{8}$$

where w_1^* characterizes the line of fixed points to which the initial barrier hierarchy $\{w_n\}$ is attracted. It is indicated that the anomalous long-time behaviour of the autocorrelation function is dependent on the maximum ramification of the fractal Koch curve.

Now we turn to the particular case of $w_j = C^j$ $(0 < C < 1,\ j = 1, 2, \ldots)$, where C is an effective temperature parameter (Huberman and Kerszbery 1985, Maritan and Stella 1986b). One can find

$$x = \begin{cases} \dfrac{2\ln[(M^2+M+2)/2]}{\ln[M(M^2+M+2)/(M-1)]} & C < (M-1)/2M \\[2ex] \dfrac{2\ln[(M^2+M+2)/2]}{\ln[(M^2+M+2)/2C]} & (M-1)/2M < C < \frac{1}{2} \\[2ex] \dfrac{2\ln[(M^2+M+2)/2]}{\ln(M^2+M+2)} & C > \frac{1}{2}. \end{cases} \tag{9}$$

Equation (9) shows that as the effective temperature parameter C is increased, two crossovers emerge: one is non-universal $C_1 = (M-1)/2M$ (it depends on M) for the transition from universal to non-universal anomalous diffusion, the other is universal $C_2 = \frac{1}{2}$ for the transition from non-universal to universal anomalous diffusion. Another interesting consequence which can be verified from equation (9) is that, if M is large enough, x will be independent of C (e.g. when $M = 50$, $x \approx 0.82$ for all C).

To summarize, we have introduced and investigated, by an exact renormalization group method, an ultradiffusion model defined on a family of fractal branching Koch curves. The non-universal time scaling exponent x of the autocorrelation function $P_0(t)$ has been found. It is indicated that the anomalous long-time behaviour of the autocorrelation function depends on the maximum ramification number of the fractal Koch curve. For the particular hierarchy we found a non-universal crossover $C_1 = (M-1)/2M$ and a universal one $C_2 = \frac{1}{2}$ as the effective temperature C is increased. We also found that for large M, the dynamic behaviour of ultradiffusion on the fractal branching Koch curves is independent of the effective temperature.

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